

The Łojasiewicz Exponent of Semiquasihomogeneous Singularities^{*†}

SZYMON BRZOSTOWSKI

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ABSTRACT

Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a semiquasihomogeneous function. We give a formula for the local Łojasiewicz exponent $\mathcal{L}_0(f)$ of f , in terms of weights of f . In particular, in the case of a quasihomogeneous isolated singularity f , we generalize a formula for $\mathcal{L}_0(f)$ of KRASIŃSKI, OLESIK and PŁOSKI ([KOP09]) from 3 to n dimensions. This was previously announced in [TYZ10], but as a matter of fact it has not been proved correctly there, as noticed by the AMS reviewer T. KRASIŃSKI.

As a consequence of our result, we get that the Łojasiewicz exponent is invariant in topologically trivial families of singularities coming from a quasihomogeneous germ. This is an affirmative partial answer to Teissier's conjecture.

1 Introduction

The (local) Łojasiewicz exponent $l_0(f)$ of a holomorphic map-germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^m, 0)$ is one of the possible generalizations of the order function from 1 to n indeterminates. Namely, for $n = 1$ the condition $\text{ord } f := \inf\{\text{ord } f_i\} = k$ is equivalent to the condition

$$C_1|z|^k \leq \|f(z)\| \leq C_2|z|^k, \quad (1)$$

for some positive constants C_1, C_2 , in a neighbourhood of 0. (Here $k < \infty$ exactly when $f \neq 0$ i.e. when f is a finite mapping.) For $n > 1$, the above condition splits: the optimal exponent in the right inequality (that is the biggest one) leads to the ordinary notion of the order $\text{ord } f$ of f while the optimal exponent in the left inequality (that is the smallest one) is the Łojasiewicz exponent $l_0(f)$ of f . (Note that $l_0(f) < \infty$ exactly when f is a finite mapping, unlike the order function.) To state it more formally,

$$l_0(f) := \inf\{q \in \mathbb{Q}_{>0} : C_1\|z\|^q \leq \|f(z)\|, \text{ for some } C_1 > 0 \text{ and all } \|z\| \ll 1\}.$$

The norms $\|\cdot\|$ in the definition are any convenient ones. An important (and, at first, maybe a little surprising) fact is that $l_0(f)$, if finite, is a positive *rational* number. Indeed, one can prove that in such a case the defining *infimum* is in fact the *minimum*. Moreover, there exist holomorphic curves $0 \neq \varphi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$ satisfying

$$l_0(f) = \frac{\text{ord } f \circ \varphi}{\text{ord } \varphi}, \quad (2)$$

and such curves are optimal in the sense that for every $0 \neq \psi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$ it holds $l_0(f) \geq \frac{\text{ord } f \circ \psi}{\text{ord } \psi}$. (For the proofs of these facts, see [LT08] or [Pł08].) It easily follows that, for the optimal curves φ ,

$$\|f \circ \varphi\| \sim \|\varphi\|^{l_0(f)},$$

so in a way we recover the inequalities (1) that we started from.

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Much is known about $l_0(f)$ for $n = 2$ (see [LT08, KL77, Tei77, CK88, Plo88, CK95, Len98, Plo01]), the case $n \geq 3$ is however challenging still (for some recent results see e.g. [Biv09, KOP09, Ole09, Plo10, BKO12, BE12]).

In this paper we are interested in the *Łojasiewicz exponent of a singularity* $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$,

$$\mathcal{L}_0(f) := l_0\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right). \quad (3)$$

More specifically, we give formulas for $\mathcal{L}_0(f)$ if f is a (weakly) (semi-)quasihomogeneous singularity (see Definition 1), in terms of its weights. Such formulas are known when $n \leq 3$ ([KOP09] for the quasihomogeneous, and [BKO12] for the semiquasihomogeneous case). In [TYZ10], there appeared an *incorrect* proof of an analogous formula for general n (cf. AMS review MR2679619 for the details). We aim at giving a valid proof of this result (Theorem 3).

Our approach to the problem follows closely that of [KOP09]: first we prove a general lemma which is interesting in its own right (Proposition 1), and then apply it to deal with the non-generic situations for the computation of the Łojasiewicz exponent. After proving Theorem 3, we pass to the more general situation allowing „weak” weights. Here, most of the necessary ingredients are delivered by SAITO ([Sai71]) whose results allow us to reduce the general problem to the semiquasihomogeneous one using stable equivalence (Corollary 3 and Theorem 4) and also express the value of the Łojasiewicz exponent of a quasihomogeneous singularity in a coordinate-independent fashion (Theorem 5). We conclude with the observation that *Teissier’s conjecture* (see at the end of the paper) is valid in the class of weakly semiquasihomogeneous functions (Corollary 5).

The main results of the paper are Theorems 3–5 and Corollary 5.

2 Definitions and Known Facts

The following definitions are much in the spirit of SAITO [Sai71] and ARNOLD [Arn74].

Definition 1 *Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a (germ of a) holomorphic function. Then:*

→ *f is an (isolated) singularity, if it has an isolated critical point at 0*

→ *f is weakly quasihomogeneous of type $(d; l_1, \dots, l_n)$, shortly: $f \in \text{WQH}(d; l_1, \dots, l_n)$, if $l = (l_1, \dots, l_n) \in \mathbb{R}^n$, $d \in \mathbb{R}_{>0}$ and for every monomial $z^a = z_1^{a_1} \cdots z_n^{a_n}$ appearing in the expansion of f with a non-zero coefficient it holds $\langle a, l \rangle := a_1 l_1 + \dots + a_n l_n = d$; in particular, $f = 0$ is WQH of all types*

The numbers l_1, \dots, l_n will be called weights. The number $\deg_l(z^a) := \langle a, l \rangle$ will be referred to as the weighted degree of a monomial z^a . For a series $g(z) = \sum_{a \in \mathbb{N}_0^n} g_a z^a$, its weighted order is $\text{ord}_l g := \inf_{g_a \neq 0} (\deg_l(z^a))$.

→ *f is quasihomogeneous of type $(d; l_1, \dots, l_n)$, shortly: $f \in \text{QH}(d; l_1, \dots, l_n)$, if it is WQH of type $(d; l_1, \dots, l_n)$ with $d, l_1, \dots, l_n \in \mathbb{Q}$ and $l_1/d, \dots, l_n/d \in (0, 1/2]$*

Note that from the definition it follows that a QH f is necessarily a (germ of a) polynomial of order greater than or equal to 2.

→ *f is weakly semiquasihomogeneous of type $(d; l_1, \dots, l_n)$, shortly: $f \in \text{WSQH}(d; l_1, \dots, l_n)$, if it can be written in the form $f = f_0 + f'$, where f_0 is a WQH singularity of type $(d; l_1, \dots, l_n)$, $\text{ord}_l f' > 1$ and every monomial appearing in the expansion of f' is of weighted degree greater than d*

The singularity f_0 will be called the principal part of f .

→ *f is semiquasihomogeneous of type $(d; l_1, \dots, l_n)$, shortly: $f \in \text{SQH}(d; l_1, \dots, l_n)$, if it is WSQH of type $(d; l_1, \dots, l_n)$ with $d, l_1, \dots, l_n \in \mathbb{Q}$ and $l_1/d, \dots, l_n/d \in (0, 1/2]$ (or f_0 is a QH singularity)*

It is known that a SQH f (or a WSQH f with positive weights) is automatically an isolated singularity [Arn74].

Remark *It is easy to see that the types $(d; l_1, \dots, l_n)$ in the definitions can always be normalized to $(1; l_1/d, \dots, l_n/d)$. Saito, op. cit., allowed also complex weights for WQH functions, however, as he proved, it is often not restrictive to consider only the rational ones. On the other hand, Arnold, op. cit., considered mostly SQH functions; the definition of WSQH functions is perhaps somewhat non-standard.*

The following theorem holds [KOP09, Thm. 1, Cor. 4 and Thm. 3].

Theorem 1 *Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$, where $n \leq 3$, be a weakly quasihomogeneous singularity of type $(1; l_1, \dots, l_n)$ with positive rational weights. Put $w_i := 1/l_i$. Then*

$$\mathcal{L}_0(f) = \min \left(\max_{1 \leq i \leq n} (w_i - 1), \prod_{1 \leq i \leq n} (w_i - 1) \right). \quad (4)$$

In particular, if f is quasihomogeneous

$$\mathcal{L}_0(f) = \max_{1 \leq i \leq n} (w_i - 1).$$

Remark Actually, formula (4) is proved in [KOP09] only for $n = 3$. However, for a function f of 2 indeterminates one can consider the function $\tilde{f} := f + z_3^2$, which has the same Łojasiewicz exponent as f and for which the weight $l_3 = 1/2$, and then apply formula (4) to it to find an analogous formula for $\mathcal{L}_0(f)$.

Theorem 1 is known to generalize to the case of a SQH function f ([BKO12, Theorem 3.2]) in exactly the same form. Namely, taking account of the remark above, one can state:

Theorem 2 *Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$, where $n \leq 3$, be a weakly semiquasihomogeneous function with positive rational weights and principal part f_0 . Then*

$$\mathcal{L}_0(f) = \mathcal{L}_0(f_0).$$

3 Results

We begin with a proof of a proposition which is a weaker version of [KOP09, Thm. 2], but generalized to $n \geq 4$ indeterminates. We remark that in *loc. cit.* the theorem is stated as a very special case of local Hilbert's Nullstellensatz and the authors conjecture it to be true in any dimension (*op. cit.*, Problem 1). It is however not the case, already for four indeterminates (cf. Example 1).

Notation For a germ $f \in \mathcal{O}^n$ of n indeterminates and $i \in \{1, \dots, n\}$ we define $\nabla f := \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right)$ and $\hat{\nabla} f := \left(\frac{\partial f}{\partial z_1}, \dots, \widehat{\frac{\partial f}{\partial z_i}}, \dots, \frac{\partial f}{\partial z_n} \right)$, where the hat means omission. For a set F of germs, $\mathcal{V}(F)$ will denote the germ at 0 of the set of common zeroes of the system F .

Proposition 1 *Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be an isolated singularity such that $\mathcal{V}(\hat{\nabla} f) \subset \mathcal{V}(z_1)$. Then $\text{ord } f = 2$. Moreover, there exists an $1 < i \leq n$ such that the monomial $z_1 z_i$ does appear in the expansion of f with a non-zero coefficient while the monomial z_1^2 does not.*

Proof Let us consider the deformation $f_s(z) := f(z) + s z_1^2$ of the germ f . For any $s \in \mathbb{C}$, it is $\mathcal{V}(\hat{\nabla} f_s) = \mathcal{V}(\hat{\nabla} f) =: \mathcal{A}$ because in fact $(\hat{\nabla} f_s) \mathbb{C}\{z_1, \dots, z_n\} = (\hat{\nabla} f) \mathbb{C}\{z_1, \dots, z_n\}$ as ideals. Take a set Φ of non-equivalent parametrizations of the curve \mathcal{A} . Then one has

$$\mu(f_s) = \sum_{\varphi \in \Phi} l_\varphi \text{ord} \left(\frac{\partial f_s}{\partial z_1} \circ \varphi \right),$$

where the numbers l_φ are the multiplicities of the branches φ of the curve \mathcal{A} . But by assumption, $\varphi \subset \mathcal{V}(\hat{\nabla} f) \subset \mathcal{V}(z_1)$ and hence $\text{ord} \left(\frac{\partial f_s}{\partial z_1} \circ \varphi \right) = \text{ord} \left(\frac{\partial f}{\partial z_1} \circ \varphi \right)$, for every $\varphi \in \Phi$. Thus, one has $\mu(f_s) = \mu(f)$, $s \in \mathbb{C}$, or in another words $-(f_s)$ is a μ -constant deformation of f . Using a result of TROTMAN (see [Tro80] or [PT12, Prop. 1.1]), we conclude that the family (f_s) is equimultiple. Since f is a singularity, $\text{ord } f_s = 2$ for s close to 0, and hence also $\text{ord } f = 2$.

Let $q(z)$ be the quadratic form of f . Assume, to the contrary, that the form does not depend on z_1 . Then, by the splitting lemma, f can be transformed through a biholomorphic change of coordinates Ψ into $\tilde{f} = \tilde{h}(z_1, \dots, z_k) + z_{k+1}^2 + \dots + z_n^2$, where $\text{ord } \tilde{h} \geq 3$ and $k \geq 1$. Moreover, it is easy to see that Ψ can be chosen so that $\Psi(z) = (z_1, \dots)$ (by assumption; just recall that Ψ engages essentially at most those variables that appear in the form $q(z)$). However, such change of parameters does not drag the zero set of $\hat{\nabla} f$ out of the hyperplane $z_1 = 0$, i.e. $\mathcal{V}(\hat{\nabla} \tilde{f}) \subset \mathcal{V}(z_1)$. Thus also $\mathcal{V}(\hat{\nabla} \tilde{h}) \subset \mathcal{V}(z_1)$, which is impossible by what we already know. It follows that q does depend on z_1 . More precisely, since every $f - \vartheta z_1^2$, $\vartheta \in \mathbb{C}$, also fulfills the conditions of the proposition, we deduce that in the expansion of f there has to appear a monomial of the form $z_1 z_i$, $i \neq 1$. If for every such i , in $q(z)$ there appeared also the monomial z_i^2 , then we would be able to transform the function f into one that does not contain z_1 in its quadratic form, possibly except for z_1^2 , but still one that fulfills the assumptions of the proposition; contradiction. \square

Remark For f (semi)quasihomogeneous, instead of Trotman's theorem one can use the results of [Gre86] or [O'S87] to prove that $\text{ord } f = 2$.

Example 1 Although we will not prove it, we remark that using Proposition 1 it is possible to show that every singularity $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ satisfying $\mathcal{V}(\hat{\nabla} f) \subset \mathcal{V}(z_1)$ can be transformed into the form $f = z_1 z_2 + g(z_2, \dots, z_n) + h(z_1)$ by a formal change of coordinates whose first component is the identity. Thus, in order to prove that under the assumptions of Proposition 1 it holds $z_1 \in (\hat{\nabla} f)$ (cf. [KOP09, Problem 1]), it is enough to check this for singularities of the form indicated above. Now, for $n = 3$ it turns out that g can be further transformed into one that does not depend on z_2 , which gives an alternative proof of *op. cit.* Theorem 2. For $n \geq 4$, let us consider any singularity $g_0 = g_0(z_3, \dots, z_n)$ such that $g_0 \notin \left(\frac{\partial g_0}{\partial z_3}, \dots, \frac{\partial g_0}{\partial z_n} \right)$, i.e. a g_0 that is not quasihomogeneous in any system of coordinates, and put $f := z_1 z_2 + (1 + z_2) g_0$. Assume, to the contrary, that $z_1 \in (\hat{\nabla} f) \mathbb{C}\{z_1, \dots, z_n\}$. It is easy to see that there has to exist a relation of the form $z_1 = \frac{\partial f}{\partial z_2} + A_3 \frac{\partial f}{\partial z_3} + \dots + A_n \frac{\partial f}{\partial z_n}$, where $A_j \in \mathbb{C}\{z_2, \dots, z_n\}$. But this relation implies that $g_0 \in \left(\frac{\partial f}{\partial z_3}, \dots, \frac{\partial f}{\partial z_n} \right) \mathbb{C}\{z_2, \dots, z_n\}$ and hence – that also $g_0 \in \left(\frac{\partial g_0}{\partial z_3}, \dots, \frac{\partial g_0}{\partial z_n} \right) \mathbb{C}\{z_3, \dots, z_n\}$, contradiction. As a more specific example, one can consider for instance $f := z_1 z_2 + (1 + z_2)(z_3^4 + z_3^2 z_4^3 + z_4^5)$. It can be checked, using a computer algebra system, that $z_1 \notin (\nabla f)$ but $z_1^2 \in (\hat{\nabla} f)$.

We suspect that it may be the case that $\mathcal{V}(\hat{\nabla} f) \subset \mathcal{V}(z_1)$ for a singularity f implies $z_1^{n-2} \in (\hat{\nabla} f)$, for $n \geq 3$.

Using Proposition 1 we easily deduce the following.

Corollary 1 For every semiquasihomogeneous function f of type $(d; l_1, \dots, l_n)$ such that $0 < l_j/d < 1/2$, $j = 1, \dots, n$, and every $i \in \{1, \dots, n\}$ it is

$$\mathcal{V}(\hat{\nabla} f) \not\subset \mathcal{V}(z_i).$$

Proof Since then f is of order greater than 2. \square

Corollary 2 Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a quasihomogeneous singularity of type $(1; l_1, \dots, l_n)$. Assume that $l_1 \leq \dots \leq l_n$ and $\mathcal{V}(\hat{\nabla} f) \subset \mathcal{V}(z_1)$. Then f is a homogeneous polynomial of order 2. In particular, $\mathcal{L}_0(f) = 1$.

Proof By Proposition 1, in f there appears a monomial $z_1 z_i$ with a non-zero coefficient. It follows that $l_1 = l_i = 1/2$ and hence also all the other weights are equal to $1/2$. Now we can apply [Plo85, Lemme 2.4] to conclude that $\mathcal{L}_0(f) = \max_{1 \leq j \leq n} \left(\text{ord } \frac{\partial f}{\partial z_j} \right) = 1$. \square

Theorem 3 Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a semiquasihomogeneous function of type $(1; l_1, \dots, l_n)$. Put $w_i := 1/l_i$. Then

$$\mathcal{L}_0(f) = \max_{1 \leq i \leq n} (w_i - 1). \quad (5)$$

Proof First assume that f is quasihomogeneous. Let $l_1 \leq \dots \leq l_n$. If $\mathcal{V}(\hat{\nabla} f) \subset \mathcal{V}(z_1)$ then formula (5) is valid by Corollary 2. In the opposite case, it is enough to apply [KOP09, Proposition 2].

For f semiquasihomogeneous, Corollary 4.8 of [BE12] or Proposition 4.1 of [BKO12] assert that $\mathcal{L}_0(f_0) \leq \mathcal{L}_0(f)$, where f_0 is the principal part of f . In another words, $\max_{1 \leq i \leq n} (w_i - 1) \leq \mathcal{L}_0(f)$. By [Plo85, Proposition 2.2] we obtain the opposite inequality. \square

It remains to consider the case of „weak weights”. For this purpose, we adopt the results of [Sai71] to WSQH functions. First, we prove a WSQH variant of SPLITTING LEMMA.

Lemma 1 *Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a weakly semiquasihomogeneous function of type $(1; l) := (1; \underbrace{p_1, \dots, p_i}_x, \underbrace{q_1, \dots, q_k, r_1, \dots, r_i}_y)$, where $2i + k = n$, and of the form*

$$f(x, y, z) = (f_0(y) + \sum_{j=1}^i x_j z_j) + f'(x, y, z), \quad (6)$$

where $(x, y, z) := (x_1, \dots, x_i, y_1, \dots, y_k, z_1, \dots, z_i)$ and $f_0(y) + \sum_{j=1}^i x_j z_j$ is the principal part of f . Assume that $q_1, \dots, q_k > 0$. Then either f is stably equivalent to a weakly semiquasihomogeneous singularity of type $(1; q_1, \dots, q_k)$ having $f_0(y)$ as its principal part (if $k > 0$) or is of type \mathcal{A}_1 (if $k = 0$).

Proof For $k = 0$ it is enough to apply the ordinary splitting lemma; hence in the following we will assume that $k > 0$. Also we exclude the case $i = 0$, as it is trivial.

First note that by our assumptions it is $p_j + r_j = 1$ ($j = 1, \dots, i$). Let $M := \mu(f_0) < \infty$ and let $g \in \text{WSQH}(1; l)$ be such that $\text{ord}_l(f - g) > M$ and of the form $g(x, y, z) = (f_0(y) + \sum_{j=1}^i x_j z_j) + g'(x, y, z)$, $(f_0(y) + \sum_{j=1}^i x_j z_j)$ being the principal part of g and g' being a non-zero polynomial. Recall that by Definition 1 necessarily $\text{ord}_l g' > 1$. We claim that for each $m \in \mathbb{N}$ it is possible to write g as

$$g(x, y, z) = f_0(y) + H^{(m)}(y) + \Lambda^{(m)}(x, y, z), \quad (7)$$

where $\text{ord}_l H^{(m)} > 1$,

$$\Lambda^{(m)}(x, y, z) = \sum_{1 \leq j \leq i} (x_j + \Gamma_j^{(m)}(x, y, z))(z_j + \Delta_j^{(m)}(x, y, z)) + R^{(m)}(x, y, z), \quad (8)$$

$\text{ord}_l \Gamma_j^{(m)} \geq d^{(1)} - r_j$, $\text{ord}_l \Delta_j^{(m)} \geq d^{(1)} - p_j$ ($j = 1, \dots, i$) and $\text{ord}_l R^{(m)} \geq m(d^{(1)} - 1) + 1$ with $d^{(1)} := \text{ord}_l g' \in (1, \infty)$. Moreover $H^{(m)}$, $\Lambda^{(m)}$, $\Gamma_j^{(m)}$, $\Delta_j^{(m)}$, $R^{(m)}$ are polynomials in x, y, z , vanishing at 0.

Indeed, for $m = 1$ it is enough to put $R^{(1)} := g'(x, y, z)$ and $\Lambda^{(1)} := \sum_{j=1}^i x_j z_j + R^{(1)}$ so that $\Gamma_j^{(1)} := \Delta_j^{(1)} := 0$ ($j = 1, \dots, i$) and $H^{(1)} := 0$.

Now, assuming (7) and (8) for some $m \in \mathbb{N}$, we decompose $R^{(m)}$ into *polynomials* in the following way:

$$R^{(m)} = \eta^{(m+1)}(y) + \sum_{1 \leq j \leq i} (x_j \delta_j^{(m+1)}(x, y, z) + z_j \gamma_j^{(m+1)}(x, y, z)), \quad (9)$$

where $\text{ord}_l \eta^{(m+1)} > 1$ and $x_j \delta_j^{(m+1)}$, $z_j \gamma_j^{(m+1)}$ are of weighted order greater than or equal to $\text{ord}_l R^{(m)}$. We put $H^{(m+1)} := H^{(m)} + \eta^{(m+1)}$, $\Lambda^{(m+1)} := \Lambda^{(m)} - \eta^{(m+1)}(y)$, $\Gamma_j^{(m+1)} := \Gamma_j^{(m)} + \gamma_j^{(m+1)}$ and $\Delta_j^{(m+1)} := \Delta_j^{(m)} + \delta_j^{(m+1)}$ ($j = 1, \dots, i$); clearly, these are polynomials, vanishing at 0. Moreover, $\text{ord}_l H^{(m+1)} > 1$,

$$\text{ord}_l \gamma_j^{(m+1)} \geq \text{ord}_l R^{(m)} - \text{ord}_l z_j \geq m(d^{(1)} - 1) + 1 - r_j \geq d^{(1)} - r_j, \quad (10)$$

$$\text{ord}_l \delta_j^{(m+1)} \geq \text{ord}_l R^{(m)} - \text{ord}_l x_j \geq m(d^{(1)} - 1) + 1 - p_j \geq d^{(1)} - p_j \quad (11)$$

and hence $\text{ord}_l \Gamma_j^{(m+1)} \geq d^{(1)} - r_j$, $\text{ord}_l \Delta_j^{(m+1)} \geq d^{(1)} - p_j$ ($j = 1, \dots, i$). By (8) and (9) we have

$$\begin{aligned} \Lambda^{(m+1)} &= \sum_{1 \leq j \leq i} (x_j + \Gamma_j^{(m)})(z_j + \Delta_j^{(m)}) + R^{(m)} - \eta^{(m+1)} = \\ &= \sum_{1 \leq j \leq i} (x_j + \Gamma_j^{(m)})(z_j + \Delta_j^{(m)}) + \sum_{1 \leq j \leq i} (x_j \delta_j^{(m+1)} + z_j \gamma_j^{(m+1)}) = \\ &= \sum_{1 \leq j \leq i} (x_j + \Gamma_j^{(m+1)})(z_j + \Delta_j^{(m+1)}) + R^{(m+1)}, \end{aligned}$$

where $R^{(m+1)} := -\sum_{1 \leq j \leq i} (\Gamma_j^{(m)} \delta_j^{(m+1)} + \gamma_j^{(m+1)} \Delta_j^{(m+1)})$. Using (11) and induction hypothesis, $\text{ord}_l \Gamma_j^{(m)} \delta_j^{(m+1)} = \text{ord}_l \Gamma_j^{(m)} + \text{ord}_l \delta_j^{(m+1)} \geq (d^{(1)} - r_j) + (m(d^{(1)} - 1) + 1 - p_j) = (m+1)(d^{(1)} - 1) + 1$ ($j = 1, \dots, i$) and similarly for the other terms. Hence also $\text{ord}_l R^{(m+1)} \geq (m+1)(d^{(1)} - 1) + 1$. Thus, (7) and (8) hold for $m+1$ and – by induction – for all $m \in \mathbb{N}$.

Fix any $m \in \mathbb{N}$ and consider $\tilde{g}_m := g - R^{(m)}$. Since $\text{ord}_l \Gamma_j^{(m)} \geq d^{(1)} - r_j > 1 - r_j = p_j = \deg_l x_j$ and similarly $\text{ord}_l \Delta_j^{(m)} \geq d^{(1)} - p_j > \deg_l z_j$ for $j = 1, \dots, i$, putting $\Psi(x, y, z) := (x_1 + \Gamma_1^{(m)}, \dots, x_i + \Gamma_i^{(m)}, y, z_1 + \Delta_1^{(m)}, \dots, z_i + \Delta_i^{(m)})$ we easily see that $\Psi(0) = 0$ and the matrix $\frac{\partial \Psi}{\partial (x, y, z)} \Big|_{(x, y, z)=0}$ is, up to permutation of rows, a triangular one. Composing \tilde{g}_m with Ψ^{-1} and then with a linear transformation, we find by (7) and (8) that \tilde{g}_m goes into the form

$$\check{g}_m(x, y, z) := f_0(y) + H^{(m)}(y) + \sum_{j=1}^i (x_j^2 + z_j^2). \quad (12)$$

Since (by our assumptions and Definition 1) f_0 is a wQH singularity of type $(1; q) := (1; q_1, \dots, q_k)$ and $\text{ord}_q H^{(m)} = \text{ord}_l H^{(m)} > 1$, we deduce that \check{g}_m is wSQH of type $(1; \frac{1}{2}, \dots, \frac{1}{2}, q_1, \dots, q_k, \frac{1}{2}, \dots, \frac{1}{2})$, and since all these weights are positive, we conclude that \check{g}_m is a singularity. More precisely, $\mu(\check{g}_m) = \mu(f_0) = M < \infty$ (by the very same reasoning as in [Arn74, Theorem 3.1] and by the stable-equivalence-invariance of the Milnor number). Hence, the degree of (right) determinacy of \check{g}_m – and thus also of \tilde{g}_m – can be bounded from above by M .

Now recall that $\text{ord}_l R^{(m)} \geq m(d^{(1)} - 1) + 1 \xrightarrow{m \rightarrow \infty} \infty$, because $d^{(1)} > 1$. Since there are only finitely many monomials of given (ordinary) degree, it follows that $\lim_{m \rightarrow \infty} \text{ord}_l R^{(m)} = \infty$. Hence, there exists an $m_0 \in \mathbb{N}$ for which the order of $R^{(m_0)}$ is higher than the number M . \tilde{g}_{m_0} being M -determined, this means that \tilde{g}_{m_0} is biholomorphically equivalent to $g = \tilde{g}_{m_0} + R^{(m_0)}$ and thus we conclude that also g is M -determined and biholomorphically equivalent to \tilde{g}_{m_0} . But in such a case f is right equivalent to g and hence – to \tilde{g}_{m_0} . Finally, from (12) it follows that f is stably equivalent to the wSQH singularity $f_0(y) + H^{(m_0)}(y)$ of type $(1; q)$ and of principal part equal to f_0 . The lemma is proved. \square

Corollary 3 *Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a weakly semiquasihomogeneous function of type $(1; l_1, \dots, l_n)$. Let $l_1 \leq \dots \leq l_i \leq 0 < l_{i+1} \leq \dots \leq l_{i+k} < 1 \leq l_{i+k+1} \leq \dots \leq l_n$. Then f is a singularity and is either stably equivalent to a weakly semiquasihomogeneous function of type $(1; l_{i+1}, \dots, l_{i+k})$ if $k \neq 0$, or is of type \mathcal{A}_1 if $k = 0$.*

Proof By [Sai71, Korollar 1.9], $i = n - i - k$ and moreover $l_j + l_{n+1-j} = 1$, for $1 \leq j \leq i$. Repeating the proof of [Sai71, Lemma 1.10] one concludes that the principal part f_0 of f can be written in the following form

$$f_0(x, y, z) = \tilde{f}_0(y) + \sum_{j=1}^i (g_j(z) + h_j(x, y, z))x_{i+1-j}, \quad (13)$$

where the coordinates in $(\mathbb{C}^n, 0)$ are denoted by $(x, y, z) := (x_1, \dots, x_i, y_{i+1}, \dots, y_{i+k}, z_{i+k+1}, \dots, z_n)$, the map-germ (g_1, \dots, g_i) is a biholomorphism of $(\mathbb{C}^i, 0)$ and the functions h_j satisfy $h_j(0, 0, z) = 0$. It follows that $G(x, y, z) := (x, y, g_1(z) + h_1(x, y, z), \dots, g_i(z) + h_i(x, y, z))$ is a biholomorphism of $(\mathbb{C}^n, 0)$. Moreover, (13) implies that each component G_j of G is wQH of type $(l_j; l_1, \dots, l_n)$, for $j = 1, \dots, i$. Using the identity $\text{Id} = G^{-1} \circ G$ we easily check that each component G_j^{-1} of G^{-1} is also wQH of type $(l_j; l_1, \dots, l_n)$, for $j = 1, \dots, i$. Hence, for every monomial w the function $(G^{-1})^* w$ is wQH of type $(\deg_l w; l_1, \dots, l_n)$. It follows that $\tilde{f} := (G^{-1})^* f$ is wSQH of type $(1; l_1, \dots, l_n)$. Writing $f = f_0 + f'$, we have

$$\tilde{f}(x, y, z) = (G^{-1})^* f(x, y, z) = (\tilde{f}_0(y) + \sum_{j=1}^i z_{i+k+j} x_{i+1-j}) + (G^{-1})^* f'(x, y, z).$$

Since the weights l_{i+1}, \dots, l_{i+k} are positive, we can apply Lemma 1 to \tilde{f} . Clearly, this gives the required assertions also for $f = G^* \tilde{f}$. \square

Corollary 4 Let $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ be a weakly semiquasihomogeneous function of type $(1; l_1, l_2, l_3)$ with rational weights. Put $w_i := \begin{cases} 1/l_i, & \text{if } l_i \notin \{0, 1\} \\ 2, & \text{if } l_i \in \{0, 1\} \end{cases}$. Then formula (4) holds with $n = 3$.

Proof Let $l_1 \leq l_2 \leq l_3$. Assume that $l_1 \leq 0$. Again by [Sai71, Korollar 1.9] we conclude that $l_3 = 1 - l_1$ and $0 < l_2 < 1$. Corollary 3 asserts that f is stably equivalent to a wSQH function \tilde{f} of type $(1; l_2)$. Hence, $0 < l_2 \leq \frac{1}{2}$ and $\mathcal{L}_0(\tilde{f}) = \text{ord } \tilde{f} - 1 = \frac{1}{l_2} - 1$. Since the Łojasiewicz exponent is an invariant of the stable equivalence, $\mathcal{L}_0(f) = \frac{1}{l_2} - 1$. We easily check that (4) has the same value. The case of all weights being positive is covered by Theorems 1 and 2. \square

We remark that the above corollary is not true for $n = 2$ unless one modifies formula (4) slightly. Similarly, it has been shown by M. Sękowski [Sęk10] that formula (4) of Theorem 1 fails in dimensions $n \geq 4$, even for wQH singularities with positive rational weights. A version of formula (4), valid in any dimension, is given below, in Theorem 4.

We illustrate Corollary 4 with the following example.

Example 2 Let us consider $f(x, y, z) = xy + x^4y^3 + (z + y)^3$. The singularity f is wSQH of type $(1; -2, 3, 1/3)$ with principal part $f_0 = xy + x^4y^3 + z^3$ and also it is wSQH of type $(1; 2/3, 1/3, 1/3)$ with principal part $\tilde{f}_0 = xy + (z + y)^3$. Corollary 4 asserts that the Łojasiewicz exponent of f is equal to 2, in both cases.

Theorem 4 Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a weakly semiquasihomogeneous function of type $(1; l_1, \dots, l_n)$. Define the multisets $L := [l_1, \dots, l_n]$, $L^- := [1 - a : a \in L \wedge a > \frac{1}{2}]$ and the set $L^0 := (L \setminus L^-) \cup \{\frac{1}{2}\}$. Put $l_{\min} := \min L^0$. Then $\mathcal{L}_0(f) = \frac{1}{l_{\min}} - 1$.

Proof By [Sai71, Korollar 1.9] the number of the weights ≤ 0 is the same as the number of the weights ≥ 1 and such weights are distributed symmetrically with respect to $1/2$, counting multiplicities. Hence, upon applying Corollary 3, we can assume that $0 < l_j < 1$, $j = 1, \dots, n$. (This affects neither $\mathcal{L}_0(f)$ nor the number l_{\min} , also in the degenerate case of all the weights lying outside the interval $(0, 1)$.)

Now we repeat the last part of the proof of [Sai71, Satz 1.3]. Let $0 < l_1 \leq \dots \leq l_n < 1$ and assume that $l_n > \frac{1}{2}$. This means that the principal part $f_0(z)$ of $f(z)$ can depend only linearly on z_n . However, f_0 is a singularity and hence its expansion has to involve a monomial of the form $z_n z_i$, for some $i < n$ (cf. [Sai71, Korollar 1.6]). It is easy to see that f_0 can be brought to the form $\tilde{f}_0(z_1, \dots, \hat{z}_i, \dots, \hat{z}_n) + z_i z_n$ by a biholomorphism that does not violate the wSQH type of f . Hence, by this biholomorphism, f is transformed to a wSQH function \tilde{f} with principal part of the form $\tilde{f}_0(z_1, \dots, \hat{z}_i, \dots, \hat{z}_n) + z_i z_n$, which is the one assumed in (6). Applying Lemma 1 to \tilde{f} we can reduce it to $(\tilde{f}_0 + \tilde{f}') (z_1, \dots, \hat{z}_i, \dots, \hat{z}_n) + z_i^2 + z_n^2$, where $\tilde{f}_0 + \tilde{f}'$ is a wSQH function of type $(1; l_1, \dots, \hat{l}_i, \dots, l_{n-1}, \hat{l}_n)$ or the zero function for $n = 2$, in which case $\mathcal{L}_0(f) = 1 = \frac{1}{l_{\min}} - 1$. In the former case, we easily see that the replacement of f with $\tilde{f} + \tilde{f}'$ again does not affect the numbers $\mathcal{L}_0(f)$ and l_{\min} . Thus we reduce the number of variables. Continuing in this way, either we end up with a sum of squares at some stage, in which case $\mathcal{L}_0(f) = 1 = \frac{1}{l_{\min}} - 1$, or with a wSQH function \tilde{f} with weights lying in the interval $(0, \frac{1}{2}]$. It is easy to see that such weights have to be rational, so in this last case \tilde{f} is in fact a SQH function. Moreover, for such weights the multiset L^- is empty and l_{\min} is just the minimal of the weights. By formula (5), we get the desired equality. \square

Commentary If f_0 is the principal part of a wSQH function f , Theorem 4 implies that $\mathcal{L}_0(f) = \mathcal{L}_0(f_0) < \infty$. This once again signifies that a wSQH function f is automatically an isolated singularity, which – by the above proof and Corollary 3 – is stably (and even biholomorphically) equivalent to a SQH function. Similarly, one can show that $\mu_0(f) = \mu_0(f_0)$.

For a wQH singularity f one can compute its Łojasiewicz exponent in any system of coordinates. Namely, we have:

Theorem 5 Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be an isolated singularity such that

$$f = g_1 \frac{\partial f}{\partial z_1} + \dots + g_n \frac{\partial f}{\partial z_n},$$

for some $g_1, \dots, g_n \in \mathbb{C}\{z_1, \dots, z_n\}$. Let $[\alpha_1, \dots, \alpha_n] \subset \mathbb{C}$ be the multiset of all eigenvalues of the matrix $\left. \frac{\partial(g_1, \dots, g_n)}{\partial(z_1, \dots, z_n)} \right|_{z=0}$. Put $l_j := \operatorname{Re} \alpha_j$, $j = 1, \dots, n$, and define L, L^-, L^0, l_{\min} as in Theorem 4. Then $\mathcal{L}_0(f) = \frac{1}{l_{\min}} - 1$.

Proof Since f is a singularity, [Sai71, Lemma 4.2] implies that $g_1(0) = \dots = g_n(0) = 0$. Next, [Sai71, Korollar 3.3] asserts that there exists a formal system of coordinates in which f is a WQH formal singularity \tilde{f} of type $(1; \alpha_1, \dots, \alpha_n)$. (Here we allow the weights to be complex numbers and \tilde{f} to be a formal power series.) But then, in this system of coordinates, \tilde{f} is also WQH of type $(1; l_1, \dots, l_n)$. Since \tilde{f} is finitely determined, we can assume it is a polynomial and by ARTIN's Theorem [Art68] – that it is actually biholomorphically equivalent to f . The theorem follows upon applying Theorem 4. \square

Remark Using (2) one can define the Łojasiewicz exponent also for $f \in \mathbb{C}[[z_1, \dots, z_n]]$. With this definition, all the above results on \mathcal{L}_0 are true in the formal setting.

Example 3 Let $f := xz + xyz^2 + xy^3 + y^3z^2 + y^5 + y^2z^4 + z^8$. Using a computer algebra system one can check that $f \in (\nabla f)$ and (non-unique) eigenvalues of the Jacobian matrix of (g_1, g_2, g_3) at 0 are $[-2919603413161694054973386275881695714936.56\dots, 1/5, 2919603413161694054973386275881695714937.56\dots]$. By Theorem 5, $\mathcal{L}_0(f) = 4$. We also remark that in this case the value of $\mathcal{L}_0(f)$ can be computed using either Theorem 2 (because $f \in \text{SQH}(1; \frac{1}{2}, \frac{1}{5}, \frac{1}{2})$) or [Ole13, Thm. 1.8 (1°)], because f is Kouchnirenko non-degenerate and its Newton diagram consists of exceptional faces only (see *op. cit.* for the details).

We end the paper with a corollary concerning the conjecture of TEISSIER. Namely, just as there is the famous ZARISKI problem on multiplicity, the same question can be asked for the Łojasiewicz exponent: is it a topological invariant of a singularity? Since the question seems to be very difficult (although it is known to have the affirmative answer in case of germs of two indeterminates, see [Tei77] or [Plo01], and also for QH singularities of three indeterminates [KOP09, Corollary 2]), it is natural to ask a weaker one:

Question (Teissier's conjecture) If (f_s) is a topologically trivial deformation of a singularity f_0 , does $\mathcal{L}_0(f_0) = \mathcal{L}_0(f_s)$, for small $s \in \mathbb{C}$?

It should be noted that it is already known that Łojasiewicz exponent is lower semi-continuous in μ -constant families ([Tei77], [Plo10]).

For SQH (and also WSQH) singularities we can answer Teissier's question in the affirmative.

Corollary 5 If $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ is a weakly semiquasihomogeneous function then the Łojasiewicz exponent \mathcal{L}_0 is constant on every topologically trivial deformation of f .

Proof Assume first that f is SQH. From Theorem 3 it follows that the Łojasiewicz exponent of f is determined by its weights. On the other hand, the Theorem of VARCHENKO [Var82] implies that the weights are invariant in μ -constant deformations of f .

If we assume only that f is WSQH, then we may biholomorphically transform f into a SQH function \tilde{f} (cf. the commentary after Theorem 4). This transformation also carries any deformation f_s of f to a deformation \tilde{f}_s of \tilde{f} ; and the Milnor and Łojasiewicz numbers remain unchanged. Thus, we reduce the problem to the first case. \square

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Szymon Brzostowski
Faculty of Mathematics
and Computer Science
University of Łódź
ul. Banacha 22,
90-238 Łódź, Poland
brzosts@math.uni.lodz.pl